Conics in Polar Coordinates

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Contents

1 Conic Sections in Polar Coordinates **3**
1.1 The Focus-Directrix characterization of conics 3
1.2 Polar equations of conics 3
1.2.1 Sketching of parabola in polar coordinates 5
1.2.2 Sketching of ellipse in polar coordinates 9
1.2.3 Sketching of hyperbola in polar coordinates 16

2 Kepler's laws **23**

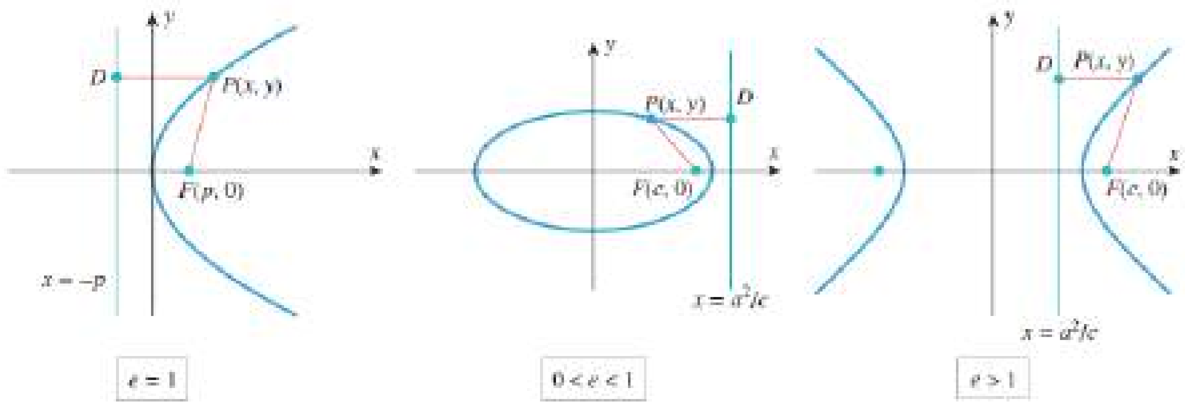
3 Orbits of planets **23**
3.1 Equation of elliptical orbits 24
3.1.1 Problems related with orbits of planets 25

1 Conic Sections in Polar Coordinates

1.1 The Focus-Directrix characterization of conics

Theorem 1.1 (Focus-Directrix Property of Conics)

Suppose that a point P moves in the plane determined by a fixed point (called the focus) and a fixed line (called the directrix), where the focus does not lie on the directrix. If the point moves in such a way that its distance to the focus divided by its distance to the directrix is some constant e (called the eccentricity), then the curve traced by the point is a conic section. Moreover, the conic is
(a) a parabola if $e = 1$ (b) an ellipse if $0 < e < 1$ (c) a hyperbola if $e > 1$.



For ellipse and hyperbola we get

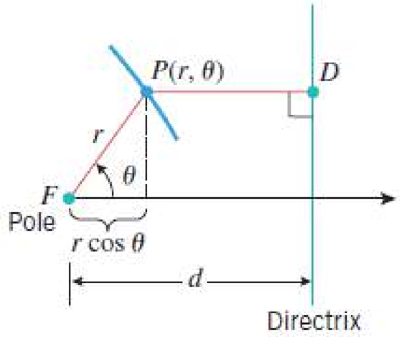
$$e = \frac{c}{a} \tag{1}$$

and the distance of directrix from the focus is

$$\frac{a^2}{c} = \frac{a^2}{ae} = \frac{a}{e} \tag{2}$$

1.2 Polar equations of conics

We derive polar equations for the conic sections from their focus-directrix characterizations. We will assume that the *focus* is at the *pole* and the *directrix* is either *parallel or perpendicular* to the *polar axis*. If the directrix is parallel to the polar axis, then it can be above or below the pole; and if the directrix is perpendicular to the polar axis, then it can be to the left or right of the pole. Thus, there are four cases to consider. We will derive the formulas for the case in which the directrix is perpendicular to the polar axis and to the right of the pole.



Assume that the directrix is perpendicular to the polar axis and d units to the right of the pole, where the constant d is known. If P is a point on the conic and if the eccentricity of the conic is e, then it follows that $PF/PD = e$, that is,

$$PF = ePD \tag{3}$$

From the figure, $PF = r, PD = d - r\cos\theta$. Thus we get from equation (3) that,

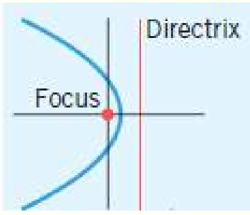
$$\begin{aligned} r &= e(d - r\cos\theta) \\ &= ed - er\cos\theta \\ \Rightarrow \qquad r + er\cos\theta &= ed \\ \Rightarrow \qquad r(1 + e\cos\theta) &= ed \\ \Rightarrow \qquad r &= \frac{ed}{1 + e\cos\theta} \end{aligned}$$

That is, general equation of a conic is

$$r = \frac{ed}{1 + e\cos\theta} \tag{4}$$

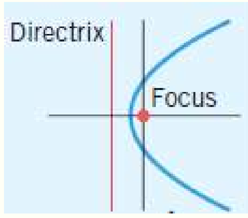
Theorem 1.2

If a conic section with eccentricity e is positioned in a polar coordinate system so that its focus is at the pole and the corresponding directrix is d units from the pole and is either parallel or perpendicular to the polar axis, then the equation of the conic has one of four possible forms, depending on its orientation:



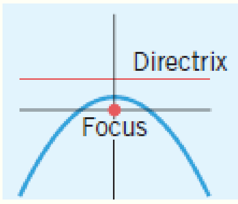
$$r = \frac{ed}{1 + e\cos\theta}$$

Directrix right of Pole



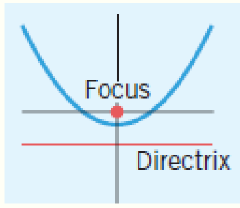
$$r = \frac{ed}{1 - e\cos\theta}$$

Directrix left of Pole



$$r = \frac{ed}{1 + e\sin\theta}$$

Directrix above Pole



$$r = \frac{ed}{1 - e\sin\theta}$$

Directrix below Pole

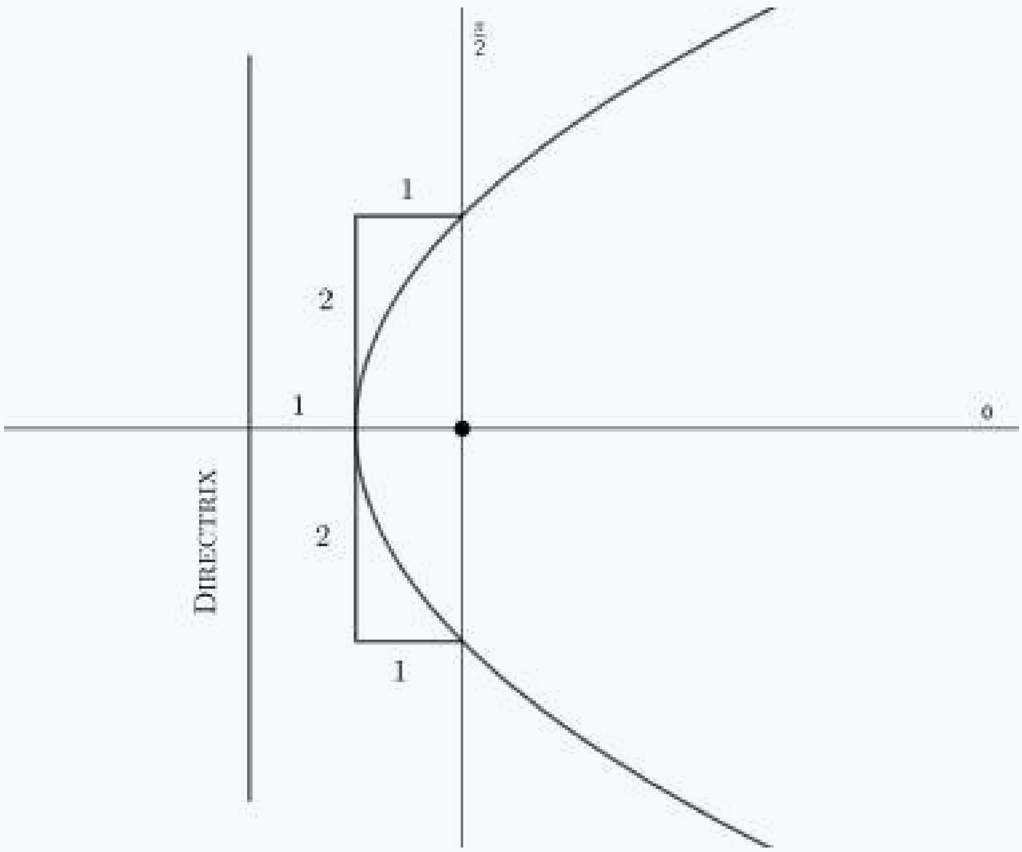
1.2.1 Sketching of parabola in polar coordinates

Problem 1.1. Sketch the graph of $r = \dfrac{2}{1 - \cos\theta}$ in polar coordinates.

Solution. The equation is similar to $r = \dfrac{ed}{1 - e\cos\theta}$, from which $ed = 2, e = 1$. That is $e = 1, d = 2$. Thus

- The conic is a parabola.
- Focus at pole.
- Directrix 2 units to the left of pole. That is, parabola opens right.
- Vertex 1 unit left of pole, so that $p = 1$.

Graph



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Problem 1.2. Sketch the graph of $r = \dfrac{6}{2 + 2\sin\theta}$ in polar coordinates.

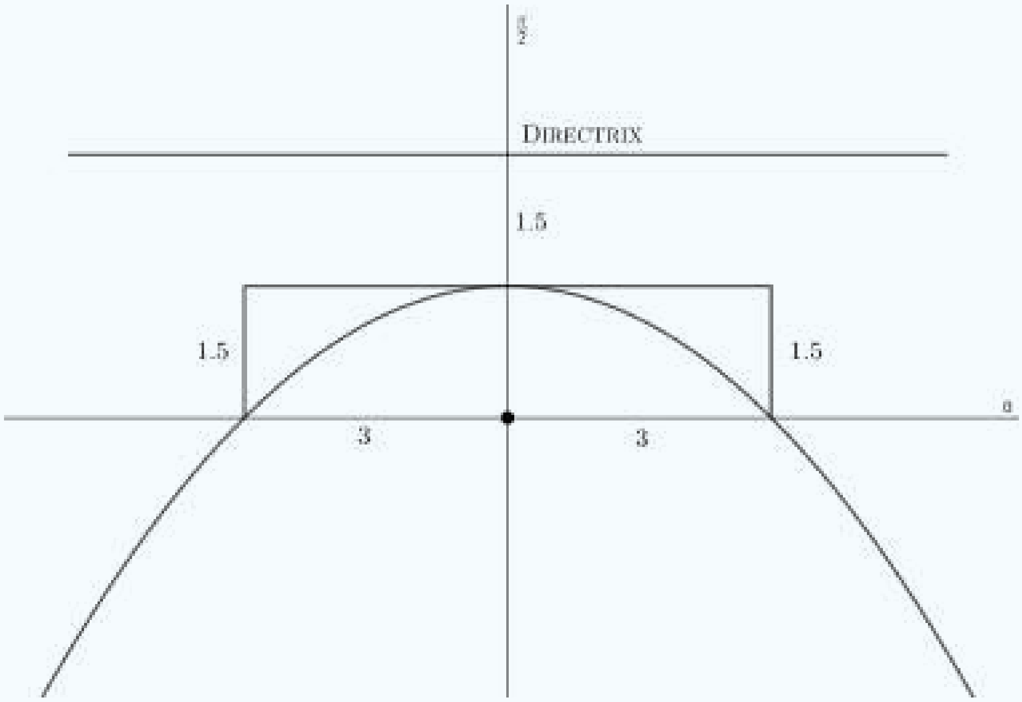
Solution. The equation can be rewritten as

$$r = \frac{3}{1 + \sin\theta}$$

which is similar to $r = \dfrac{ed}{1 + e\sin\theta}$, from which $ed = 3, e = 1$. That is $e = 1, d = 3$. Thus

- The conic is a parabola.
- Focus at pole.
- Directrix 3 units above pole. That is, parabola opens down.
- Vertex 1.5 unit above pole, so that $p = 1.5$.

Graph



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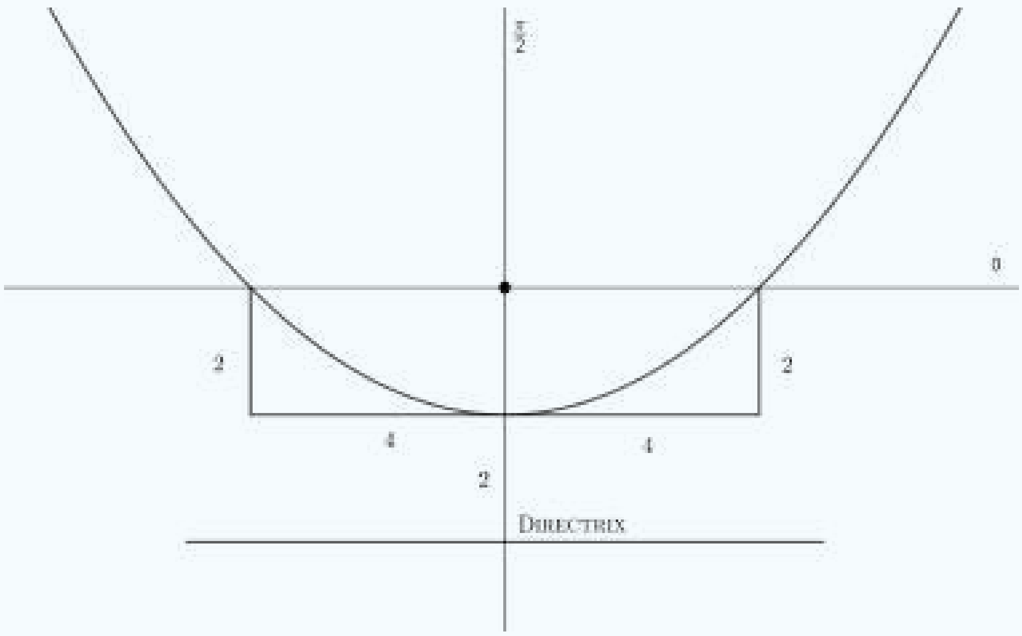
Problem 1.3. Sketch the graph of $r = \dfrac{4}{1-\sin\theta}$ in polar coordinates.

Solution. The equation is similar to $r = \dfrac{ed}{1-e\sin\theta}$, from which $ed = 4, e = 1$. That is $e = 1, d = 4$. Thus

- The conic is a parabola.
- Focus at pole.
- Directrix 4 units below pole. That is, parabola opens up.
- Vertex 2 unit below pole, so that $p = 2$.

Graph



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Problem 1.4. Sketch the graph of $r = \dfrac{3}{2+2\cos\theta}$ in polar coordinates.

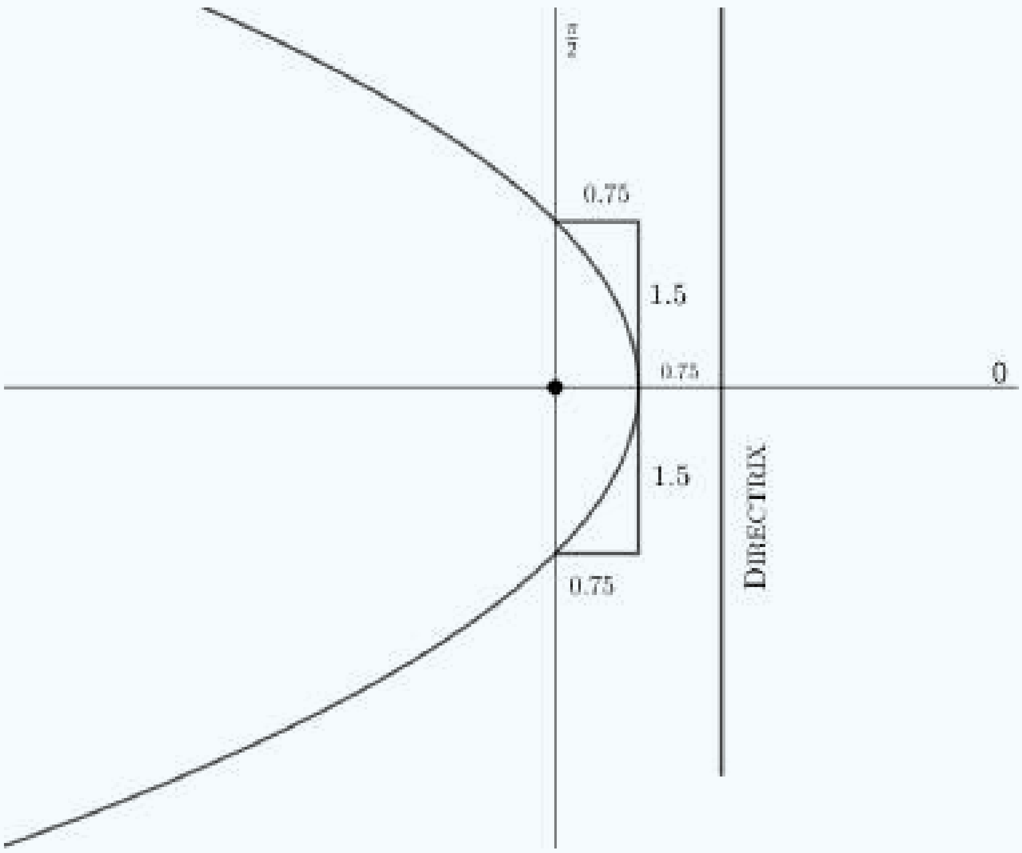
Solution. The equation can be rewritten as

$$r = \frac{1.5}{1+\cos\theta}$$

which is similar to $r = \dfrac{ed}{1+e\cos\theta}$, from which $ed = 1.5, e = 1$. That is $e = 1, d = 1.5$. Thus

- The conic is a parabola.
- Focus at pole.
- Directrix 1.5 units right of pole. That is, parabola opens left.
- Vertex 0.75 unit right of pole, so that $p = 0.75$.

Graph



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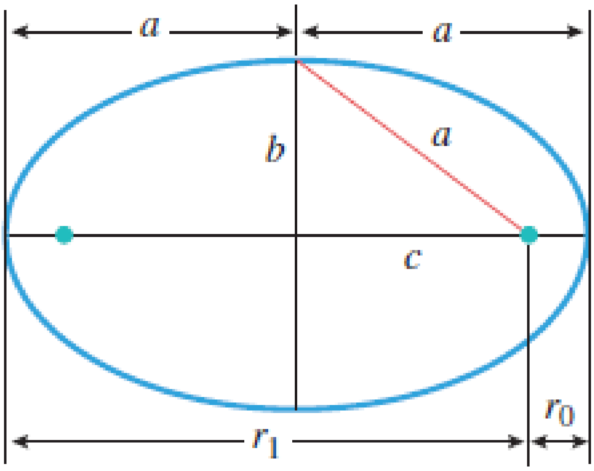
1.2.2 Sketching of ellipse in polar coordinates

For sketching an ellipse, the values of a, b, and c are to be found out. Let r_0 be the distance from the focus to the closest vertex and r_1 the distance to the farthest vertex.



Then

$$r_0 = a - c \text{ and } r_1 = a + c$$

so that,

$$
\begin{aligned}
a &= \frac{1}{2}(r_1 + r_0) \\
c &= \frac{1}{2}(r_1 - r_0)
\end{aligned}
\tag{5}
$$

Moreover,

$$r_0 r_1 = a^2 - c^2 = b^2$$

and so

$$b = \sqrt{r_0 r_1} \tag{6}$$

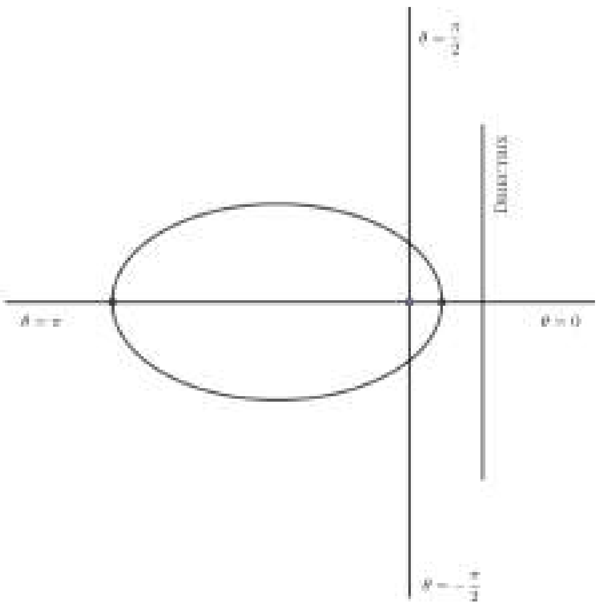
Problem 1.5. Sketch the graph of $r = \dfrac{6}{2 + \cos\theta}$ in polar coordinates.

Solution. The equation can be rewritten as

$$r = \frac{3}{1 + \frac{1}{2}\cos\theta}$$

which is similar to $r = \dfrac{ed}{1 + e\cos\theta}$, from which $ed = 3, e = \frac{1}{2}$. That is $e = \frac{1}{2}, d = 6$. Thus

- The conic is an ellipse.
- Focus at pole.
- Directrix 6 units right of pole.
- The distance r_0 from the focus to the closest vertex can be obtained by setting $\theta = 0$ in this equation, and
- the distance r_1 to the farthest vertex can be obtained by setting $\theta = \pi$



Thus,

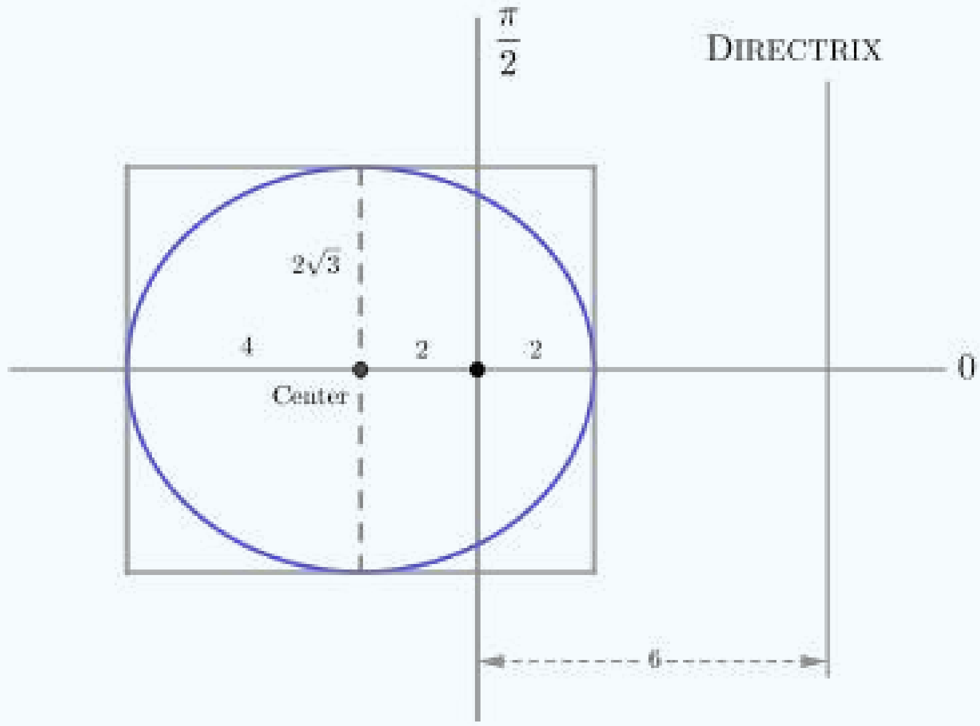
$$r_0 = \frac{3}{1+\frac{1}{2}\cos 0} = \frac{3}{1+\frac{1}{2}} = \frac{3}{\frac{3}{2}} = 2$$
$$r_1 = \frac{3}{1+\frac{1}{2}\cos \pi} = \frac{3}{1-\frac{1}{2}} = \frac{3}{\frac{1}{2}} = 6$$

Using the formulas for a, b, c, we get

$$a = \frac{1}{2}(r_1+r_0) = \frac{1}{2}(2+6) = 4, \quad b = \sqrt{r_0 r_1} = \sqrt{12} = 2\sqrt{3}, \quad c = \frac{1}{2}(r_1-r_0) = \frac{1}{2}(6-2) = 2$$

The graph of the ellipse is,

Graph



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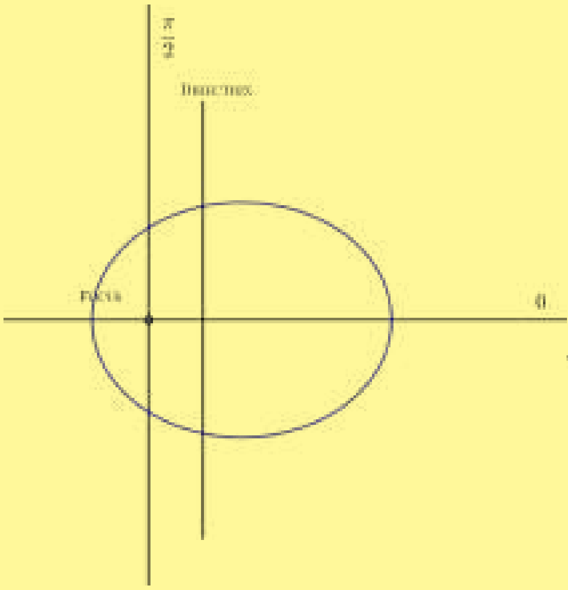
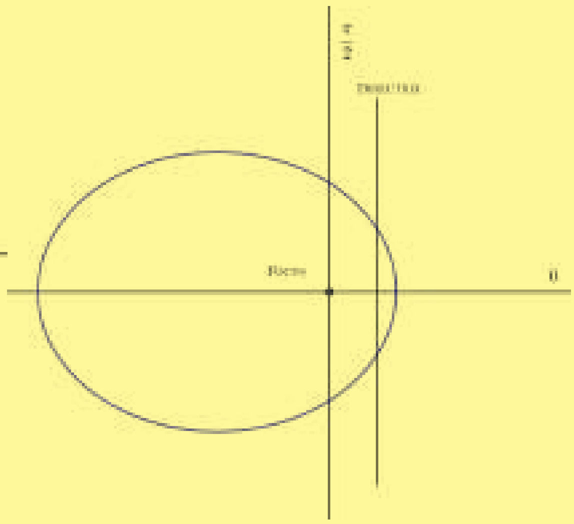
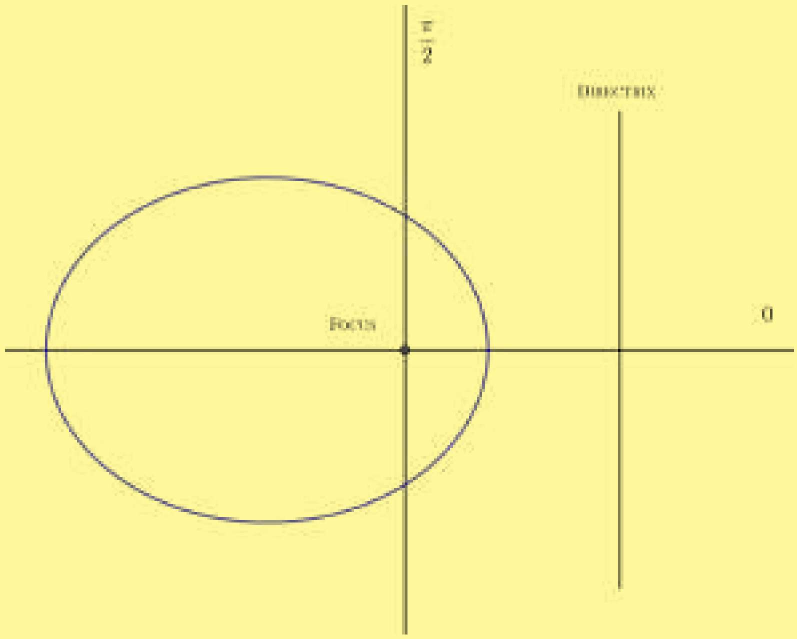
Note

While sketching the ellipse if the directrix is right of the pole there is a possibility to sketch it as





These are wrong pictures, since the directrix will not intersect the ellipse. The right sketch is



Problem 1.6. Sketch the graph of $r = \dfrac{4}{2-\cos\theta}$ in polar coordinates.

Solution. The equation can be rewritten as

$$r = \frac{2}{1-\frac{1}{2}\cos\theta}$$

which is similar to $r = \dfrac{ed}{1-e\cos\theta}$, from which $ed = 2, e = \frac{1}{2}$. That is $e = \frac{1}{2}, d = 4$. Thus

- The conic is an ellipse.

- Focus at pole.
- Directrix 4 units left of pole.
- The distance r_0 from the focus to the closest vertex can be obtained by setting $\theta = \pi$ in this equation, and
- the distance r_1 to the farthest vertex can be obtained by setting $\theta = 0$

Thus,

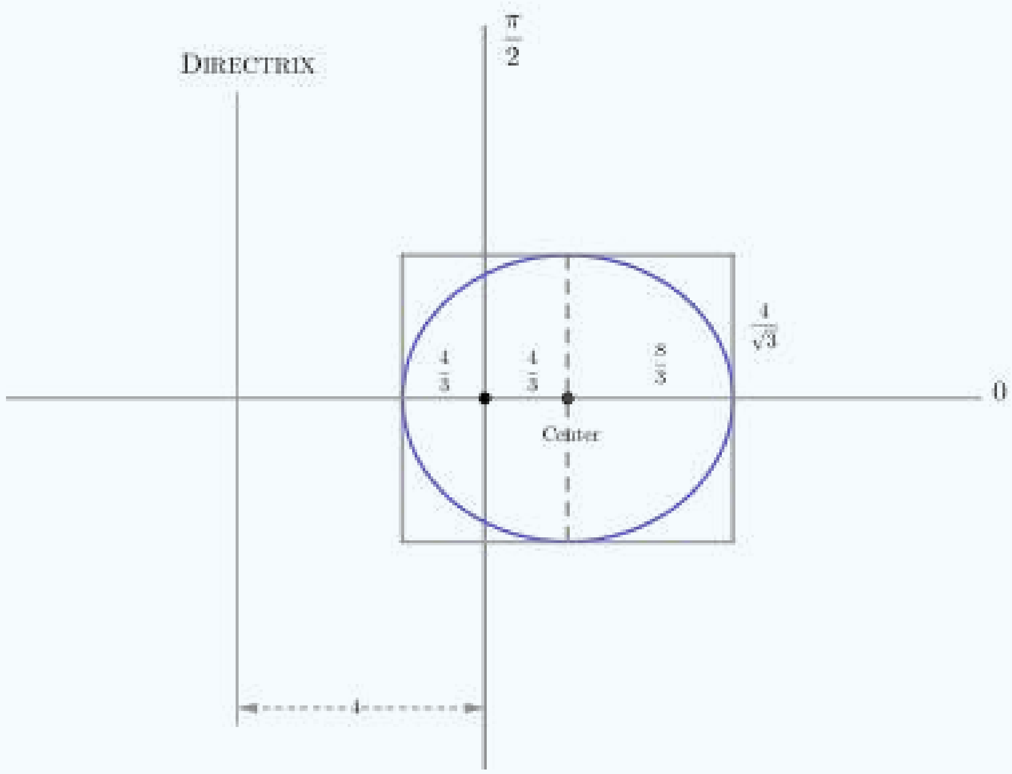
$$r_0 = \frac{2}{1 - \frac{1}{2}\cos\pi} = \frac{2}{1 + \frac{1}{2}} = \frac{2}{\frac{3}{2}} = \frac{4}{3}$$
$$r_1 = \frac{2}{1 - \frac{1}{2}\cos 0} = \frac{2}{1 - \frac{1}{2}} = \frac{2}{\frac{1}{2}} = 4$$

Using the formulas for a, b, c, we get

$$a = \frac{1}{2}(r_1 + r_0) = \frac{1}{2}(4 + \frac{4}{3}) = \frac{8}{3}, \quad b = \sqrt{r_0 r_1} = \sqrt{\frac{16}{3}} = \frac{4}{\sqrt{3}}, \quad c = \frac{1}{2}(r_1 - r_0) = \frac{1}{2}(4 - \frac{4}{3}) = \frac{4}{3}$$

The graph of the ellipse is,

Graph



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Problem 1.7. Sketch the graph of $r = \dfrac{8}{4 - 3\sin\theta}$ in polar coordinates.

Solution. The equation can be rewritten as

$$r = \frac{2}{1 - \frac{3}{4}\sin\theta}$$

which is similar to $r = \dfrac{ed}{1 - e\sin\theta}$, from which $ed = 2, e = \frac{3}{4}$. That is $e = \frac{1}{2}, d = \frac{8}{3}$. Thus

- The conic is an ellipse.
- Focus at pole.
- Directrix $\frac{8}{3}$ units below pole.
- The distance r_0 from the focus to the closest vertex can be obtained by setting $\theta = -\frac{\pi}{2}$ in this equation, and
- the distance r_1 to the farthest vertex can be obtained by setting $\theta = \frac{\pi}{2}$

Thus,

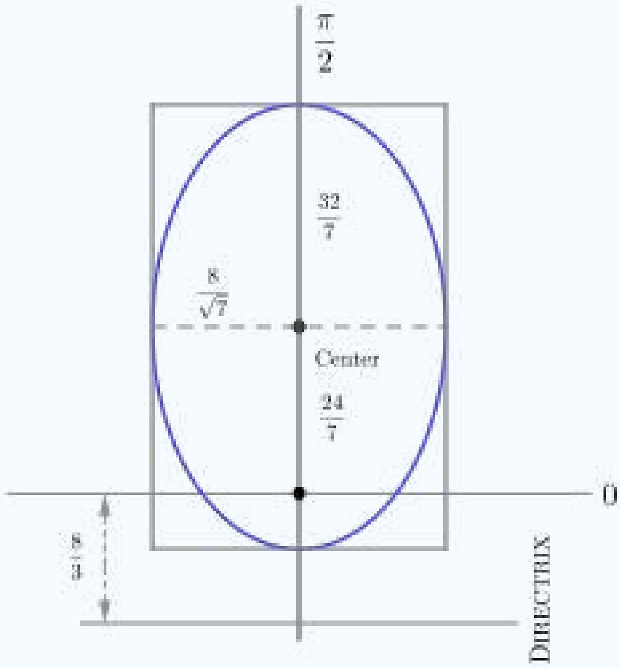
$$r_0 = \frac{2}{1 - \frac{3}{4}\sin(-\frac{\pi}{2})} = \frac{2}{1 + \frac{3}{4}} = \frac{2}{\frac{7}{4}} = \frac{8}{7}$$
$$r_1 = \frac{2}{1 - \frac{3}{4}\sin(\frac{\pi}{2})} = \frac{2}{1 - \frac{3}{4}} = \frac{2}{\frac{1}{4}} = 8$$

Using the formulas for a, b, c, we get

$$a = \frac{1}{2}(r_1 + r_0) = \frac{1}{2}(8 + \frac{8}{7}) = \frac{32}{7}, \quad b = \sqrt{r_0 r_1} = \sqrt{\frac{64}{7}} = \frac{8}{\sqrt{7}}, \quad c = \frac{1}{2}(r_1 - r_0) = \frac{1}{2}(8 - \frac{8}{7}) = \frac{24}{7}$$

The graph of the ellipse is,

Graph



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Problem 1.8. Sketch the graph of $r = \dfrac{6}{2+\sin\theta}$ in polar coordinates.

Solution. The equation can be rewritten as

$$r = \frac{3}{1+\frac{1}{2}\sin\theta}$$

which is similar to $r = \dfrac{ed}{1+e\sin\theta}$, from which $ed = 3, e = \frac{1}{2}$. That is $e = \frac{1}{2}, d = 6$. Thus

- The conic is an ellipse.
- Focus at pole.
- Directrix 6 units above pole.
- The distance r_0 from the focus to the closest vertex can be obtained by setting $\theta = \frac{\pi}{2}$ in this equation, and
- the distance r_1 to the farthest vertex can be obtained by setting $\theta = -\frac{\pi}{2}$

Thus,

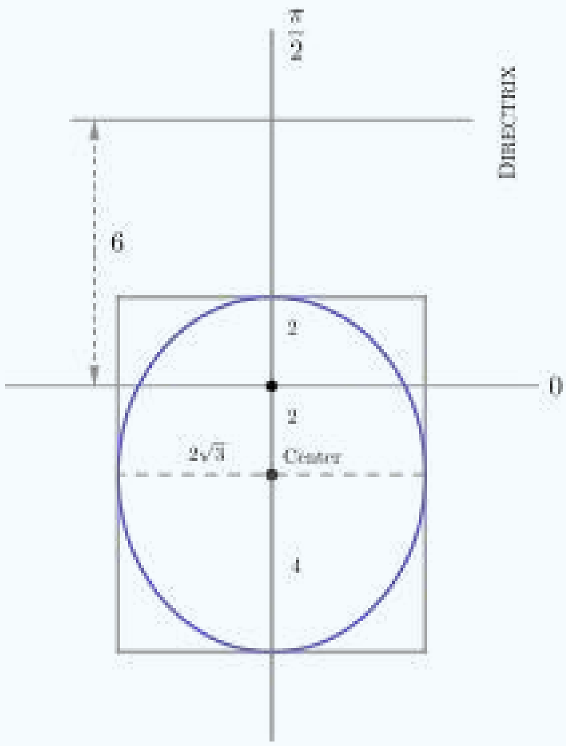
$$r_0 = \frac{3}{1+\frac{1}{2}\sin(\frac{\pi}{2})} = \frac{3}{1+\frac{1}{2}} = \frac{3}{\frac{3}{2}} = 2$$
$$r_1 = \frac{3}{1+\frac{1}{2}\sin(-\frac{\pi}{2})} = \frac{3}{1-\frac{1}{2}} = \frac{3}{\frac{1}{2}} = 6$$

Using the formulas for a, b, c, we get

$a = \dfrac{1}{2}(r_1+r_0) = \dfrac{1}{2}(6+2) = 4, \quad b = \sqrt{r_0 r_1} = \sqrt{12} = 2\sqrt{3}, \quad c = \dfrac{1}{2}(r_1-r_0) = \dfrac{1}{2}(6-2) = 2$

The graph of the ellipse is,

Graph



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The following table summarises all information about ellipse in polar coordinates

Summary Ellipse

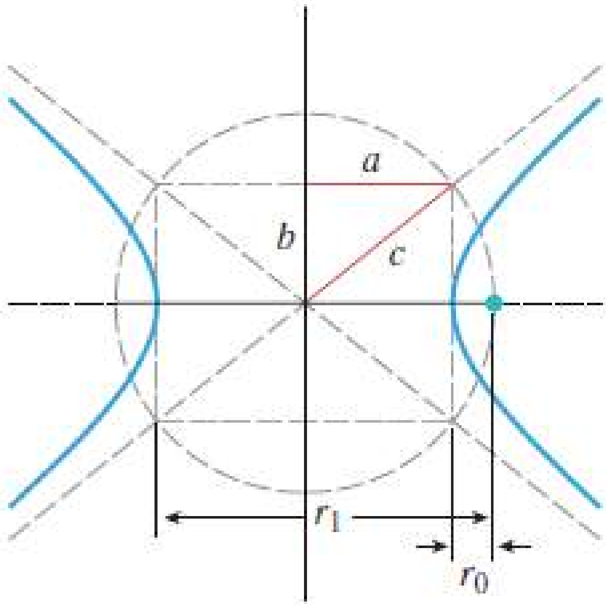
Equation	Position of Directrix	r_0	r_1
$r = \frac{ed}{1+e\cos\theta}$	Right of Pole	$\theta = 0$	$\theta = \pi$
$r = \frac{ed}{1-e\cos\theta}$	Left of Pole	$\theta = \pi$	$\theta = 0$
$r = \frac{ed}{1+e\sin\theta}$	Above Pole	$\theta = \frac{\pi}{2}$	$\theta = -\frac{\pi}{2}$
$r = \frac{ed}{1-e\sin\theta}$	Below Pole	$\theta = -\frac{\pi}{2}$	$\theta = \frac{\pi}{2}$

Note that, in all cases,

- $e < 1$.
- the directrix will never intersect the ellipse.
- the vertex closest to focus will be between focus and directrix.

1.2.3 Sketching of hyperbola in polar coordinates

For sketching a hyperbola, the values of a, b, and c are to be found out. Let r_0 be the distance from the focus to the closest vertex and r_1 the distance to the farthest vertex.



Then

$$r_0 = c - a \text{ and } r_1 = c + a$$

so that,

$$
\begin{aligned}
a &= \frac{1}{2}(r_1 - r_0) \\
c &= \frac{1}{2}(r_1 + r_0)
\end{aligned}
\tag{7}
$$

Moreover,

$$r_0 r_1 = c^2 - a^2 = b^2 \tag{8}$$

and so

$$b = \sqrt{r_0 r_1}$$

The following table summarises all information about hyperbola in polar coordinates

Summary Hyperbola

Equation	Position of Directrix	r_0 and r_1
$r = \frac{ed}{1+e\cos\theta}$ $r = \frac{ed}{1-e\cos\theta}$	Right of Pole Left of Pole	Put $\theta = 0, \theta = \pi$, find out $\lvert r \rvert$. Smallest will be r_0 and the other will be r_1
$r = \frac{ed}{1+e\sin\theta}$ $r = \frac{ed}{1-e\sin\theta}$	Above Pole Below Pole	Put $\theta = \frac{\pi}{2}, \theta = -\frac{\pi}{2}$, find out $\lvert r \rvert$. Smallest will be r_0 and the other will be r_1

Note that, in all cases,

- $e > 1$.
- the directrix will never intersect the hyperbola.
- the vertex closest to focus will be between focus and directrix.

Problem 1.9. Sketch the graph of $r = \dfrac{2}{1 + 2\sin\theta}$ in polar coordinates.

Solution. The equation is similar to $r = \dfrac{ed}{1 + e\sin\theta}$, from which $ed = 2, e = 2$. That is $e = 2, d = 1$. Thus

- The conic is a hyperbola.
- Focus at pole.
- Directrix 1 unit above pole.
- The distance r_0 from the focus to the closest vertex and the distance r_1 to the farthest vertex can be obtained by setting $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$, then taking the absolute value.

When $\theta = \frac{\pi}{2}, \theta = -\frac{\pi}{2}$

$$|r| = \left|\frac{2}{1+2\sin(\frac{\pi}{2})}\right| = \left|\frac{2}{1+2}\right| = \frac{2}{3}$$
$$|r| = \left|\frac{2}{1+2\sin(-\frac{\pi}{2})}\right| = \left|\frac{2}{1-2}\right| = 2$$

Thus

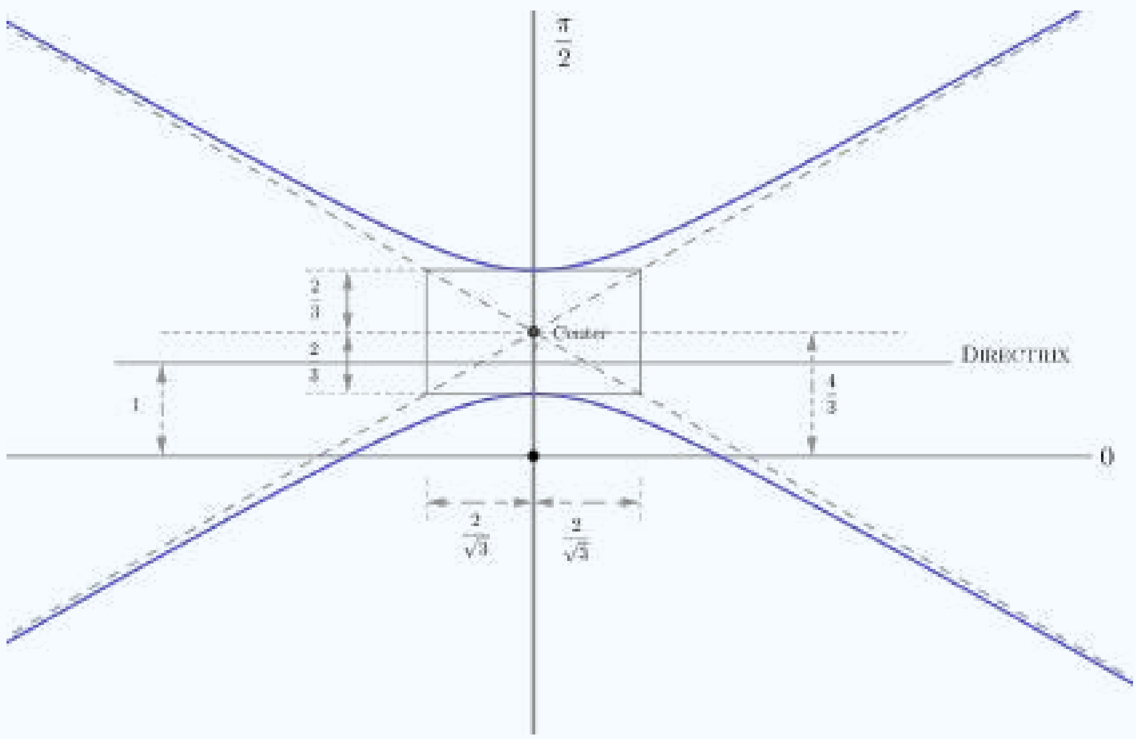
$$r_0 = \frac{2}{3}, \quad r_1 = 2.$$

Using the formulas for a, b, c, we get

$$a = \frac{1}{2}(r_1 - r_0) = \frac{1}{2}(2 - \frac{2}{3}) = \frac{2}{3}, \quad b = \sqrt{r_0 r_1} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}, \quad c = \frac{1}{2}(r_1 + r_0) = \frac{1}{2}(2 + \frac{2}{3}) = \frac{4}{3}$$

The graph of the hyperbola is,

Graph



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Problem 1.10. Sketch the graph of $r = \dfrac{4}{2+3\cos\theta}$ in polar coordinates.

Solution. The equation can be rewritten as

$$r = \frac{2}{1+\frac{3}{2}\cos\theta}$$

which is similar to $r = \dfrac{ed}{1+e\cos\theta}$, from which $ed = 2, e = \frac{3}{2}$. That is $e = \frac{3}{2}, d = \frac{4}{3}$. Thus

- The conic is a hyperbola.

- Focus at pole.
- Directrix $\frac{4}{3}$ right of pole.
- The distance r_0 from the focus to the closest vertex and the distance r_1 to the farthest vertex can be obtained by setting $\theta = 0$ and $\theta = \pi$, then taking the absolute value.

When $\theta = 0, \theta = \pi$

$$|r| = \left|\frac{2}{1+\frac{3}{2}\cos 0}\right| = \left|\frac{2}{1+\frac{3}{2}}\right| = \left|\frac{2}{\frac{5}{2}}\right| = \frac{4}{5}$$
$$|r| = \left|\frac{2}{1+\frac{3}{2}\cos \pi}\right| = \left|\frac{2}{1-\frac{3}{2}}\right| = 4$$

Thus

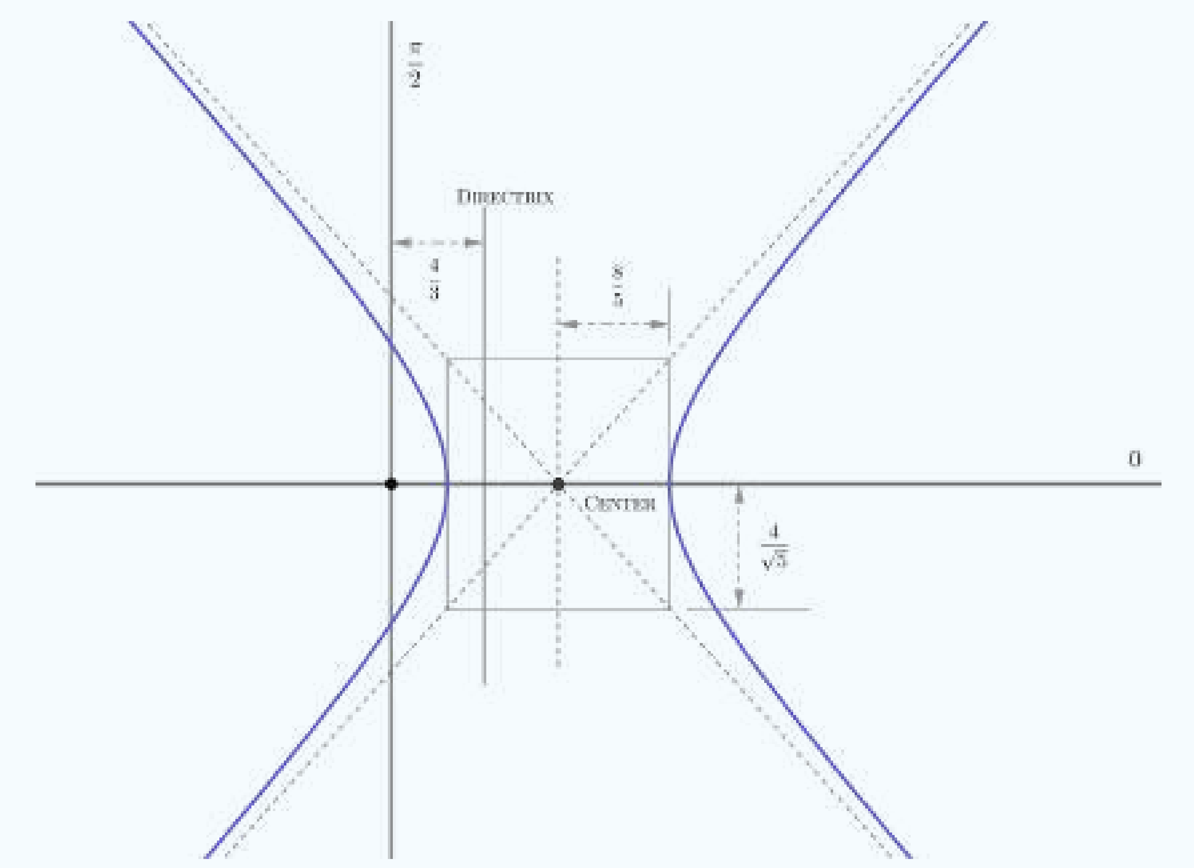
$$r_0 = \frac{4}{5} = 0.8, \quad r_1 = 4.$$

Using the formulas for a, b, c, we get

$$a = \frac{1}{2}(r_1 - r_0) = \frac{1}{2}(4-0.8) = 1.6 = \frac{8}{5},$$
$$b = \sqrt{r_0 r_1} = \sqrt{\frac{16}{5}} = \frac{4}{\sqrt{5}}$$
$$c = \frac{1}{2}(r_1 + r_0) = \frac{1}{2}(4+0.8) = 2.4 = \frac{12}{5}$$

The graph of the hyperbola is,

Graph



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Problem 1.11. Sketch the graph of $r = \dfrac{5}{2-3\cos\theta}$ in polar coordinates.

Solution. The equation can be rewritten as

$$r = \frac{\frac{5}{2}}{1-\frac{3}{2}\cos\theta}$$

which is similar to $r = \dfrac{ed}{1-e\cos\theta}$, from which $ed = \frac{5}{2}, e = \frac{3}{2}$. That is $e = \frac{3}{2}, d = \frac{5}{3}$. Thus

- The conic is a hyperbola.
- Focus at pole.
- Directrix $\frac{5}{3}$ left of pole.
- The distance r_0 from the focus to the closest vertex and the distance r_1 to the farthest vertex can be obtained by setting $\theta = 0$ and $\theta = \pi$, then taking the absolute value.

When $\theta = 0, \theta = \pi$

$$|r| = \left|\frac{\frac{5}{2}}{1-\frac{3}{2}\cos 0}\right| = \left|\frac{\frac{5}{2}}{1-\frac{3}{2}}\right| = \left|\frac{\frac{5}{2}}{-\frac{1}{2}}\right| = 5$$
$$|r| = \left|\frac{\frac{5}{2}}{1-\frac{3}{2}\cos\pi}\right| = \left|\frac{\frac{5}{2}}{1+\frac{3}{2}}\right| = \left|\frac{\frac{5}{2}}{\frac{5}{2}}\right| = 1$$

Thus

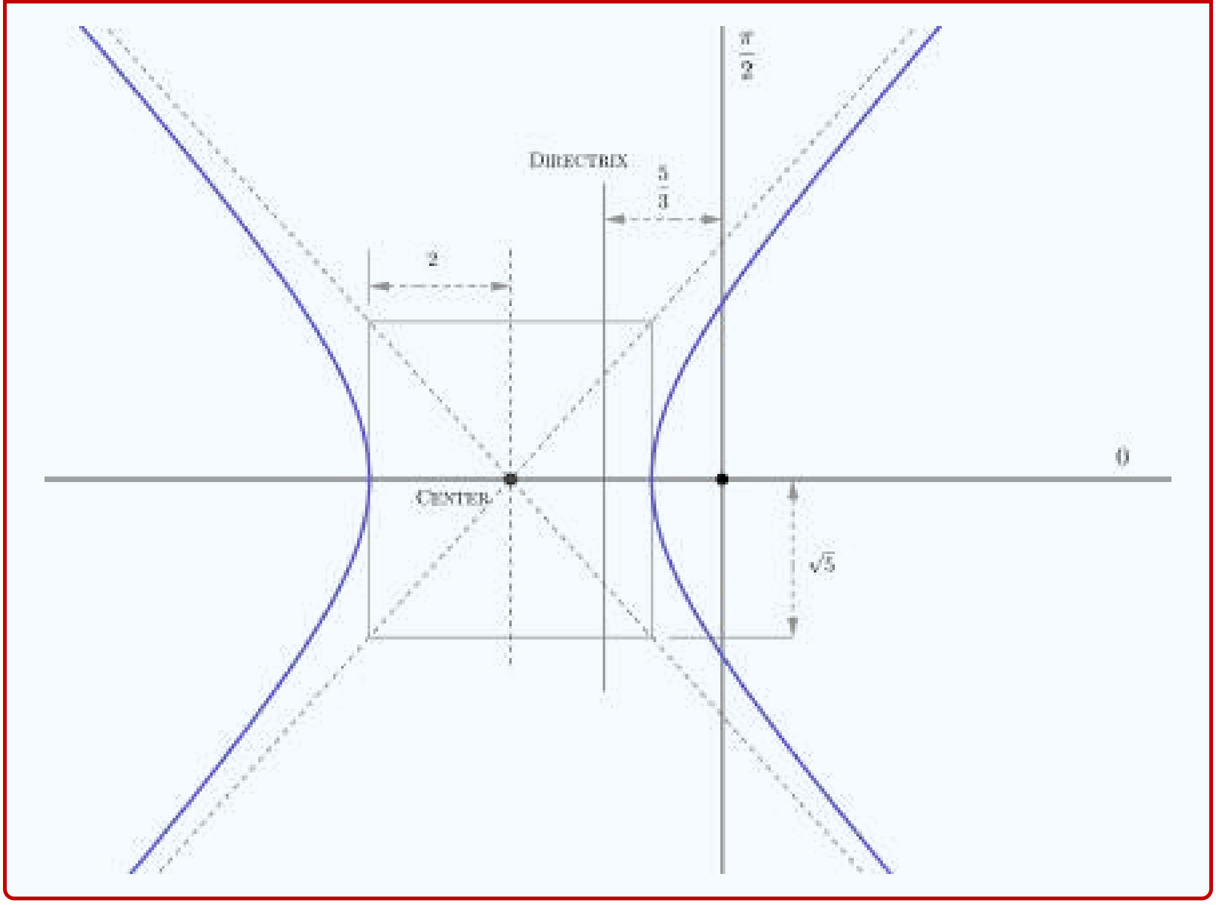
$$r_0 = 1, \quad r_1 = 5.$$

Using the formulas for a, b, c, we get

$$a = \frac{1}{2}(r_1 - r_0) = \frac{1}{2}(5-1) = 2, \quad b = \sqrt{r_0 r_1} = \sqrt{5}, \quad c = \frac{1}{2}(r_1 + r_0) = \frac{1}{2}(5+1) = 3$$

The graph of the hyperbola is,

Graph



□

Problem 1.12. Sketch the graph of $r = \dfrac{4}{1 - 2\sin\theta}$ in polar coordinates.

Solution. The equation is similar to $r = \dfrac{ed}{1 - e\sin\theta}$, from which $ed = 4, e = 2$. That is $e = 2, d = 2$. Thus

- The conic is a hyperbola.
- Focus at pole.
- Directrix 2 below pole.
- The distance r_0 from the focus to the closest vertex and the distance r_1 to the farthest vertex can be obtained by setting $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$, then taking the absolute value.

When $\theta = \frac{\pi}{2}, \theta = -\frac{\pi}{2}$

$$|r| = \left|\frac{4}{1 - 2\sin(\frac{\pi}{2})}\right| = \left|\frac{4}{1 - 2}\right| = \left|\frac{4}{-1}\right| = 4$$

$$|r| = \left|\frac{4}{1 - 2\sin(-\frac{\pi}{2})}\right| = \left|\frac{4}{1 + 2}\right| = \frac{4}{3}$$

Thus

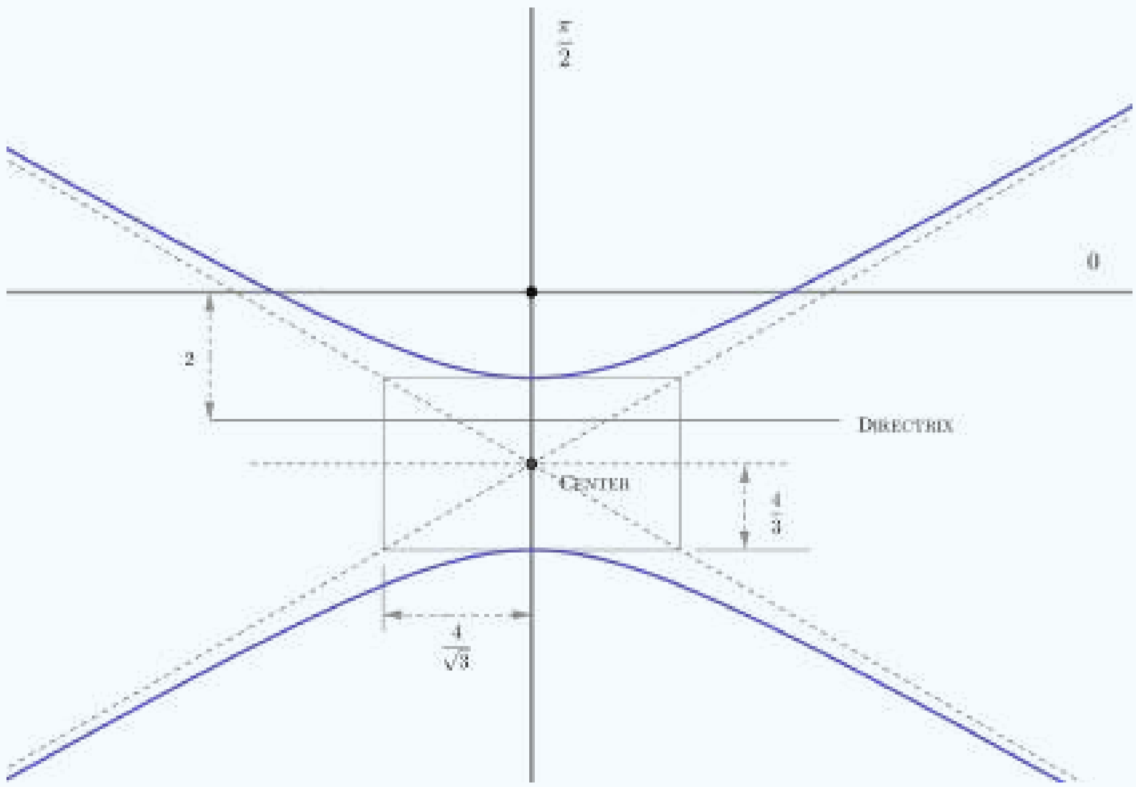
$$r_0 = \frac{4}{3}, \quad r_1 = 4.$$

Using the formulas for a, b, c, we get

$$a = \frac{1}{2}(r_1 - r_0) = \frac{1}{2}(4 - \frac{4}{3}) = \frac{4}{3}, \quad b = \sqrt{r_0 r_1} = \sqrt{\frac{16}{3}} = \frac{4}{\sqrt{3}}, \quad c = \frac{1}{2}(r_1 + r_0) = \frac{1}{2}(4 + \frac{4}{3}) = \frac{8}{3}$$

The graph of the hyperbola is,

Graph

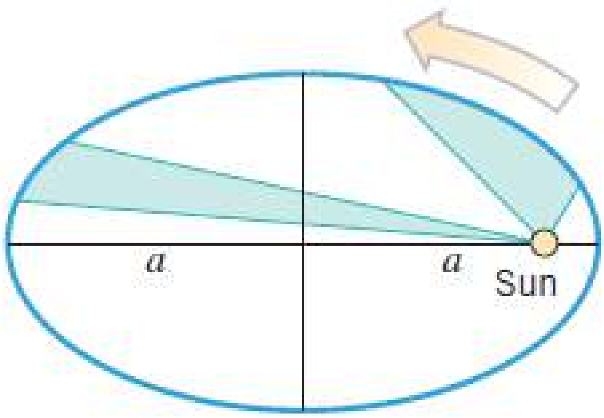


□

2 Kepler's laws

Theorem 2.3 (Kepler's laws)

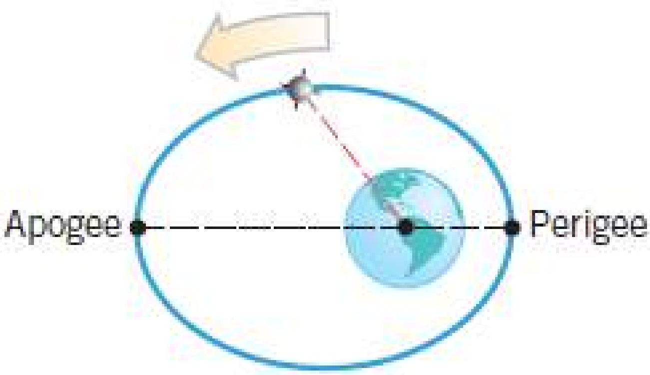
- First law (*Law of Orbits*). Each planet moves in an elliptical orbit with the Sun at a focus.
- Second law (*Law of Areas*). The radial line from the center of the Sun to the center of a planet sweeps out equal areas in equal times.
- Third law (*Law of Periods*). The square of a planet's period (the time it takes the planet to complete one orbit about the Sun) is proportional to the cube of the semimajor axis of its orbit.



Equal areas are swept out in equal times, and the square of the period T is proportional to a^3.

3 Orbits of planets

In an elliptical orbit, the closest point to the focus is called the *perigee* and the farthest point the *apogee*.



The distances from Sun (which is at the focus) to the perigee and apogee are called the *perihelion* (perigee distance) and *aphelion* (apogee distance), respectively.

Let T denotes the period of a planet and a semimajor axis of its orbit. The semimajor axis of Earth's orbit is known as 1 astronomical unit (AU) (approximately 150×10^6 km or 92.9×10^6 mi). Then for earth $a = 1$ AU and $T = 1$ earth year. By Kepler's third law, for a planet

$$T^2 \propto a^3$$

That is,

$$T^2 = ka^3$$

where k is a constant. Since $a = 1$ AU produces a period of $T = 1$ earth year (considering earth), we get

$$k = 1$$

Hence Kepler's third law can be expressed as

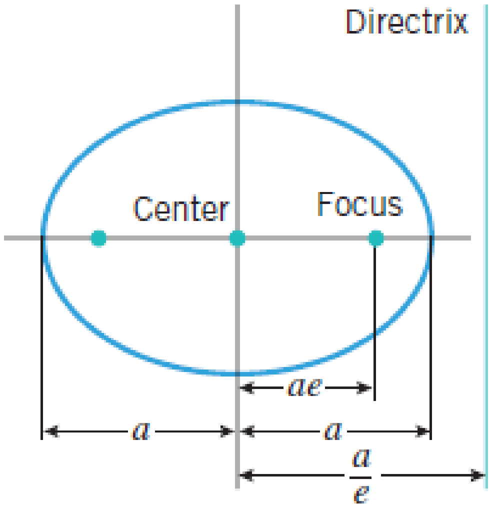
$$T^2 = a^3 \tag{9}$$

Equivalently

$$T = a^{3/2} \tag{10}$$

3.1 Equation of elliptical orbits

Polar equations of elliptical orbits are often specified by giving the eccentricity e and the semimajor axis a.



From equation (1) $c = ae$ and from equation (2), distance from center to directrix is $\frac{a}{e}$. Hence from the figure, distance d between the focus and the directrix is

$$d = \frac{a}{e} - c = \frac{a}{e} - ae = \frac{a(1-e^2)}{e}$$

That is

$$ed = a(1-e^2)$$

Hence the equation of the elliptical orbit can be expressed as

Elliptical orbit

$r = \dfrac{a(1-e^2)}{1+e\cos\theta}$ Directrix right of pole	$r = \dfrac{a(1-e^2)}{1-e\cos\theta}$ Directrix left of pole
$r = \dfrac{a(1-e^2)}{1+e\sin\theta}$ Directrix above pole	$r = \dfrac{a(1-e^2)}{1-e\sin\theta}$ Directrix below pole

Moreover, it is evident from the above figure that the distances from the focus to the closest and farthest vertices can be expressed in terms of a and e as

$$\begin{aligned} r_0 &= a - ae = a(1-e) \\ r_1 &= a + ae = a(1+e) \end{aligned} \tag{11}$$

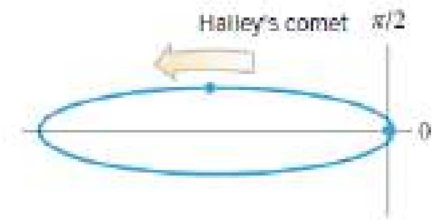
3.1.1 Problems related with orbits of planets

Problem 3.1. Halley's comet (last seen in 1986) has an eccentricity of 0.97 and a semimajor axis of $a = 18.1\,\text{AU}$.

(a) Find the equation of its orbit in the polar coordinate system as shown in the figure.

(b) Find the period of its orbit.

(c) Find its perihelion and aphelion distances.



Solution. (a) The figure shows that polar equation of the orbit has the form

$$r = \frac{a(1-e^2)}{1+e\cos\theta}.$$

But $a(1-e^2) = 18.1(1-0.97^2) \approx 1.07$. Thus, the equation of the orbit is

$$r = \frac{1.07}{1+0.97\cos\theta}.$$

(b) From (10), we have

$$T = a^{3/2} = (18.1)^{3/2} \approx 77 \text{ years}.$$

(c) Since the perihelion and aphelion distances are the distances to the closest and farthest vertices, we have

$$\begin{aligned}
\text{perihelion} = r_0 &= a(1-e) \\
&= 18.1(1-0.97) \\
&\approx 0.543\,\text{AU} \\
&= 0.543(150\times 10^6)\,\text{km} \\
&\approx 81,500,000\,\text{km} \\
\text{aphelion} = r_1 &= a(1+e) \\
&= 18.1(1+0.97) \\
&\approx 35.7\,\text{AU} \\
&= 35.7(150\times 10^6)\,\text{km} \\
&\approx 5,350,000,000\,\text{km}
\end{aligned}$$

□

Problem 3.2. The dwarf planet Pluto has eccentricity $e = 0.249$ and semimajor axis $a = 39.5AU$.

(a) Find the period T in years.

(b) Find the perihelion and aphelion distances.

(c) Choose a polar coordinate system with the center of the Sun at the pole, and find a polar equation of Pluto's orbit in that coordinate system.

(d) Make a sketch of the orbit with reasonably accurate proportions.

Solution. (a) Period, $T = a^{3/2} = 39.5^{3/2} \approx 248$ years.

(b) Perihelion $= r_0 = a(1-e) = 39.5(1-0.249)(150 \times 10^6)\,\text{km} \approx 444, 96, 75, 000\,\text{km}$.

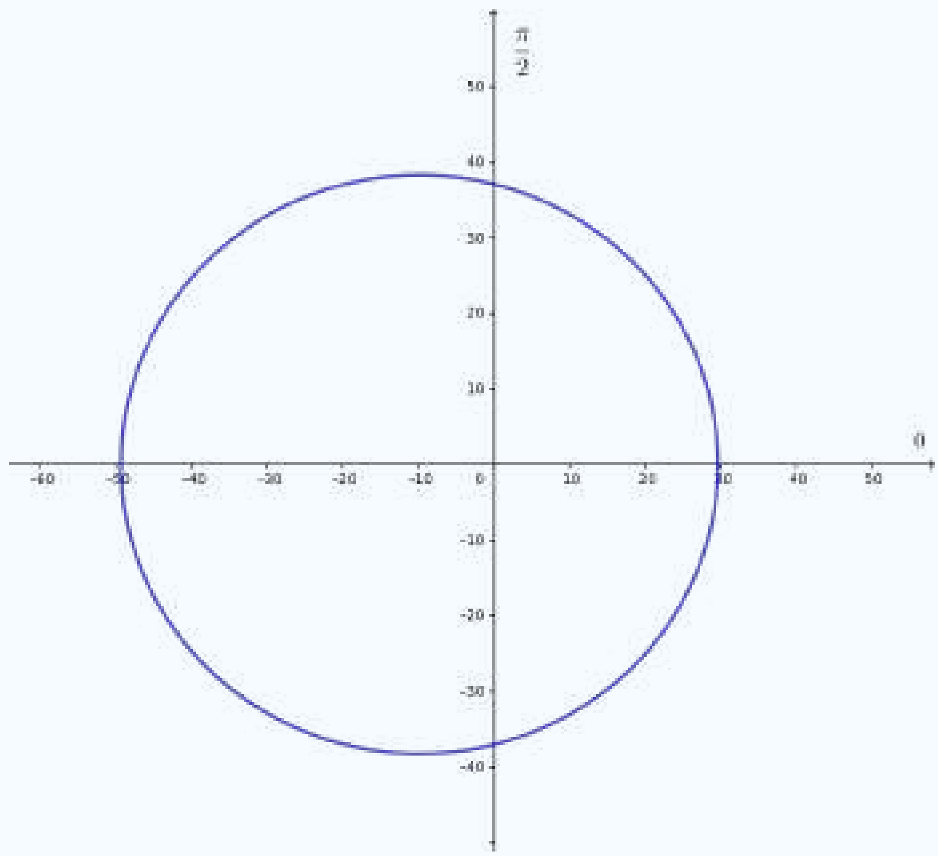
Aphelion $= r_1 = a(1+e) = 39.5(1+0.249)(150 \times 10^6)\,\text{km} \approx 740, 03, 25, 000\,\text{km}$.

(c) The equation is

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \approx= \frac{39.5(1-0.249^2)}{1+0.249\cos\theta} \approx \frac{37.05}{1+0.249\cos\theta}$$

(d)

Graph



□

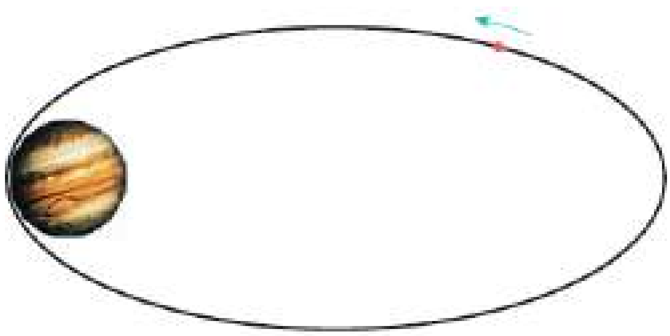
Problem 3.3. AnApollo lunar lander orbits the Moon in an elliptic orbit with eccentricity $e = 0.12$ and semimajor axis $a = 2015$ km. Assuming the Moon to be a sphere of radius 1740 km, find the minimum and maximum heights of the lander above the lunar surface.

Solution. If we let r_0 and r_1 denote the minimum and maximum distances from the center of the Moon, then the minimum and maximum distances from the surface of the Moon will be

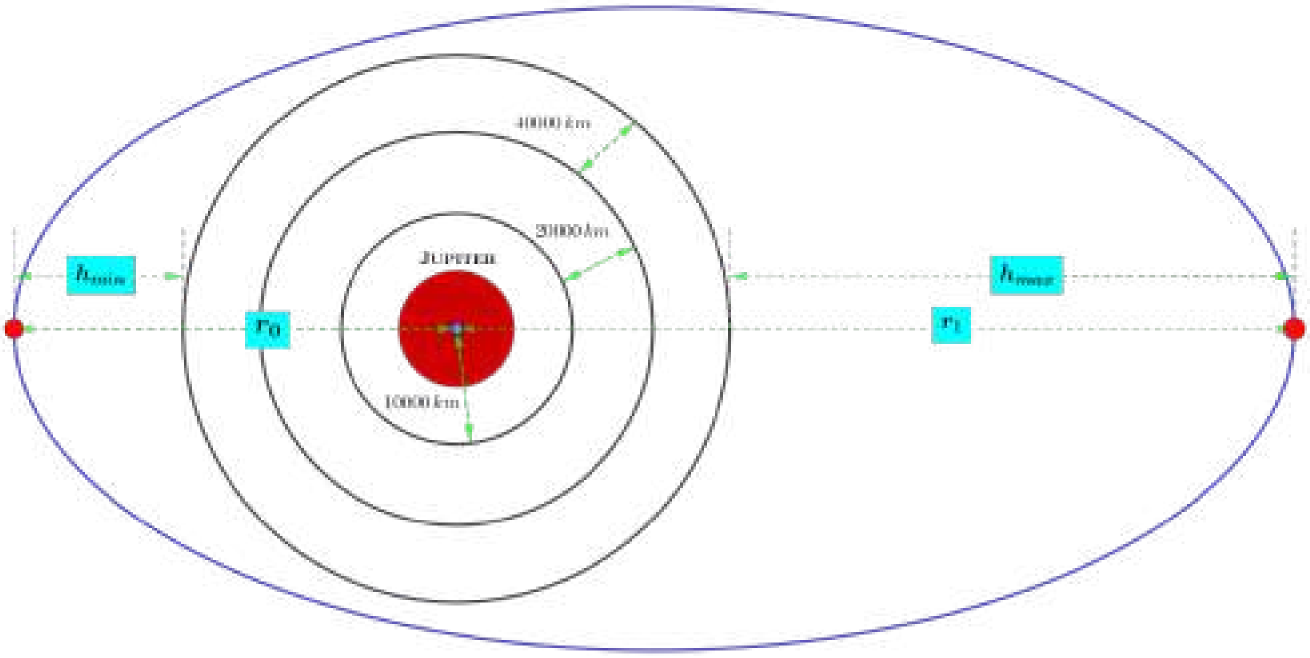
$$\begin{aligned} d_{min} &= r_0 - 1740 \\ &= a(1-e) - 1740 \\ &= 2015(0.88) - 1740 \\ &= 33.2 \text{ km} \\ d_{max} &= r_1 - 1740 \\ &= a(1+e) - 1740 \\ &= 2015(1.12) - 1740 \\ &= 516.8 \text{ km} \end{aligned}$$

□

Problem 3.4. The planet Jupiter is believed to have a rocky core of radius $10,000$ km surrounded by two layers of hydrogen - a $40,000$ km thick layer of compressed metallic-like hydrogen and a $20,000$ km thick layer of ordinary molecular hydrogen. The visible features, such as the Great Red Spot, are at the outer surface of the molecular hydrogen layer. On November 6, 1997 the spacecraft Galileo was placed in a Jovian orbit to study the moon Europa. The orbit had eccentricity 0.814580 and semimajor axis $3,514,918.9$ km. Find Galileo's minimum and maximum heights above the molecular hydrogen layer (see the accompanying figure).



Solution. The following figure gives information about Jupiter and surrounding materials.



It is clear that total thickness of rock and hydrogen layers is 70,000 km. Hence Galileo's minimum and maximum heights above Jupiter will be $h_{min} = r_0 - 70,000$ and $h_{max} = r_1 - 70,000$ respectively. Since $a = 3,514,918.9$ km and $e = 0.814580$, we have

$$r_0 = a(1-e) = 3514918.9(1-0.814580) \approx 6,51,736 \text{ km}$$
$$r_1 = a(1+e) = 3514918.9(1+0.814580) \approx 63,78,102 \text{ km}.$$

Hence minimum height of Galileo above Jupiter is

$$h_{min} = 651736 - 70000 = 5,81,736 \text{ km}$$

and maximum height of Galileo above Jupiter is

$$h_{max} = 6378102 - 70000 = 63,08,102 \text{ km}$$

□

Problem 3.5. The Hale-Bopp comet, discovered independently on July 23, 1995 by Alan Hale and Thomas Bopp, has an orbital eccentricity of $e = 0.9951$ and a period of 2380 years.

(a) Find its semimajor axis in astronomical units (AU).

(b) Find its perihelion and aphelion distances.

(c) Choose a polar coordinate system with the center of the Sun at the pole, and find an equation for the Hale-Bopp orbit in that coordinate system.

(d) Make a sketch of the Hale-Bopp orbit with reasonably accurate proportions

Solution. (a) Given $T = 2380$. So $a = T^{3/2} = 2380^{3/2} \approx 178.26\,\text{AU}$.

(b) Perihelion$= r_0 = a(1-e) = 178.26(1-0.9951)(150\times10^6)\,\text{km} \approx 13,10,21,100\,\text{km}$

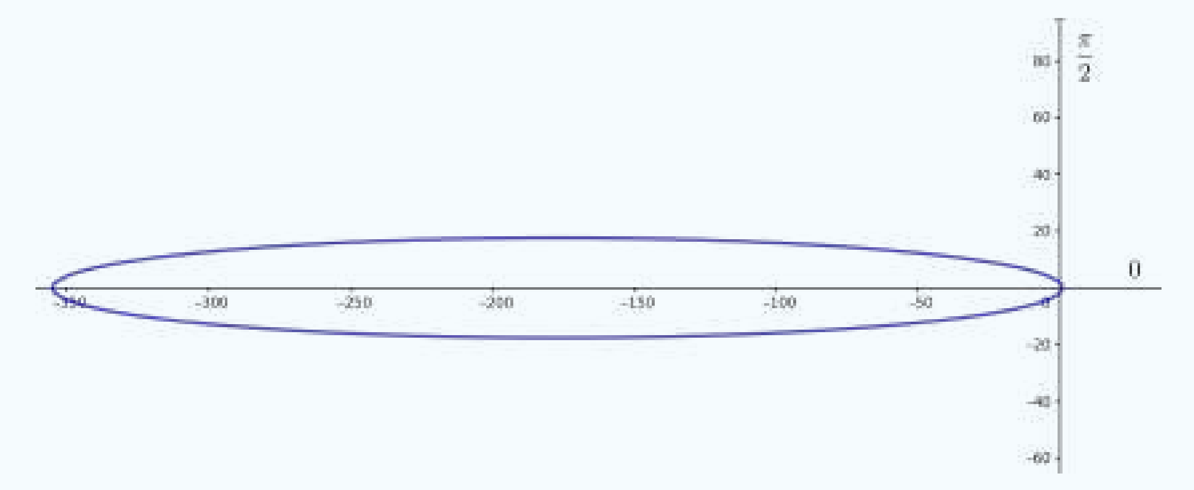
Aphelion$= r_0 = a(1+e) = 178.26(1+0.9951)(150\times10^6)\,\text{km} \approx 5334,69,78,900\,\text{km}$

(c) An equation to the orbit is

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{178.26(1-0.9951^2)}{1+0.9951\cos\theta} \approx \frac{1.74}{1+0.9951\cos\theta}$$

(d)

Graph



$\square$

Problem 3.6.

(a) Let a be the semimajor axis of a planet's orbit around the Sun, and let T be its period. Show that if T is measured in days and a is measured in kilometers, then $T = (365\times10^{-9})(a/150)^{3/2}$.

(b) Use the result in part (a) to find the period of the planet Mercury in days, given that its semimajor axis is $a = 57.95\times10^6\,\text{km}$.

(c) Choose a polar coordinate system with the Sun at the pole, and find an equation for the orbit of Mercury in that coordinate system given that the eccentricity of the orbit is $e = 0.206$.

(d) Use a graphing utility to generate the orbit of Mercury from the equation obtained in part (c).

Solution. (a) By Kepler's third law $T = a^{3/2}$ where a is measured in AU and T in Earth year. We know that

$$1\,\text{AU} = 150\times10^6\,\text{km} \quad \text{and} \quad 1\,\text{Earth year} = 365\,\text{days}$$

Converting to km and days, we get

$$T\,\text{days} = \frac{T}{365}\,\text{years} \quad \text{and} \quad a\,\text{km} = \frac{a}{150\times 10^6}\,\text{AU}$$

Hence

$$\frac{T}{365} = \left(\frac{a}{150\times 10^6}\right)^{3/2}$$
$$\Rightarrow T = 365\left(\frac{a}{150}\right)^{3/2} \times (10^{-6})^{3/2}$$
$$= (365\times(10^{-9})\left(\frac{a}{150}\right)^{3/2}$$

(b) For Mercury $a = 57.95\times 10^6$. Hence its period is

$$T = (365\times(10^{-9})\left(\frac{57.95\times 10^6}{150}\right)^{3/2} \approx 87.6\,\text{days}$$

(c) Polar equation of the orbit is

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

We have

$$a(1-e^2) = (57.95\times 10^6)(1-0.206^2)\,\text{km} = \frac{(57.95\times 10^6)(1-0.206^2)}{150\times 10^6}\,\text{AU} \approx 0.37\,\text{AU}.$$

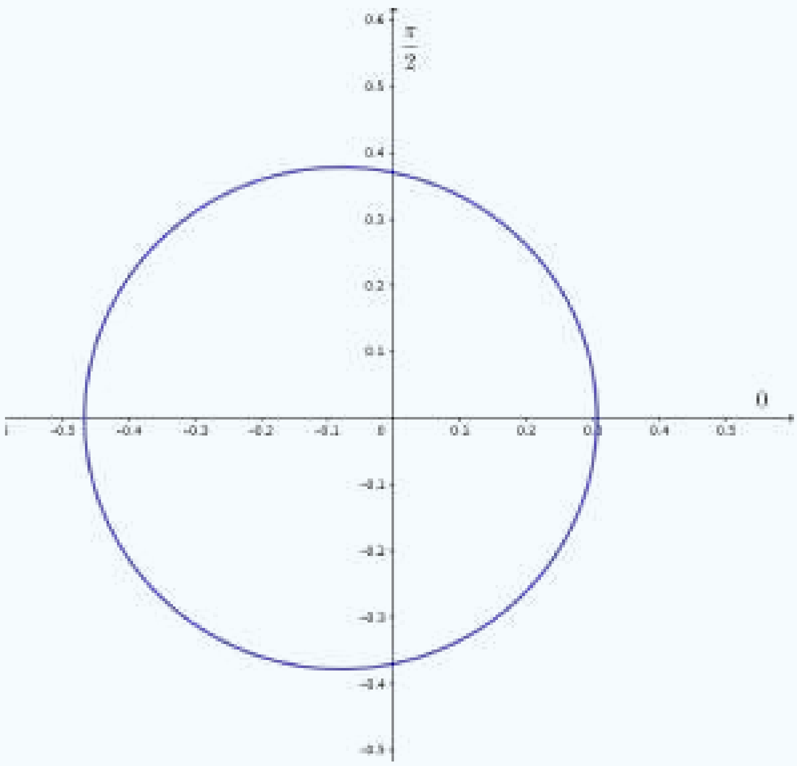
Thus

$$r = \frac{0.37}{1+0.206\cos\theta}\,\text{AU}$$

is the equation of the orbit.

(d)

Graph



$\square$

Problem 3.7.

(a) Show that the eccentricity of an ellipse can be expressed in terms of r_0 and r_1 as

$$e = \frac{r_1 - r_0}{r_1 + r_0}$$

(b) Show that

$$\frac{r_1}{r_0} = \frac{1+e}{1-e}$$

Solution. From equation (1), we have

$$e = \frac{c}{a}$$

and from equation (5)

$$a = \frac{1}{2}(r_1 + r_0) \text{ and } c = \frac{1}{2}(r_1 - r_0).$$

Hence

$$\begin{aligned} e &= \frac{c}{a} \\ &= \frac{\frac{1}{2}(r_1 - r_0)}{\frac{1}{2}(r_1 + r_0)} \\ &= \frac{r_1 - r_0}{r_1 + r_0} \end{aligned}$$

Also using the properties of proportion, we get

$$\begin{aligned} && e &= \frac{r_1 - r_0}{r_1 + r_0} \\ \Rightarrow && \frac{e}{1} &= \frac{r_1 - r_0}{r_1 + r_0} \\ \Rightarrow && \frac{1}{e} &= \frac{r_1 + r_0}{r_1 - r_0} \\ \Rightarrow && \frac{1+e}{1-e} &= \frac{(r_1 + r_0) + (r_1 - r_0)}{(r_1 + r_0) - (r_1 - r_0)} \\ && &= \frac{2r_1}{2r_0} \\ && &= \frac{r_1}{r_0} \end{aligned}$$

$\square$

(a) Show that the eccentricity of a hyperbola can be expressed in terms of r_0 and r_1 as

$$e = \frac{r_1 + r_0}{r_1 - r_0}$$

(b) Show that

$$\frac{r_1}{r_0} = \frac{e+1}{e-1}$$

Solution. From equation (1), we have

$$e = \frac{c}{a}$$

and from equation (7)

$$a = \frac{1}{2}(r_1 - r_0) \text{ and } c = \frac{1}{2}(r_1 + r_0).$$

Hence

$$\begin{aligned} e &= \frac{c}{a} \\ &= \frac{\frac{1}{2}(r_1 + r_0)}{\frac{1}{2}(r_1 - r_0)} \\ &= \frac{r_1 + r_0}{r_1 - r_0} \end{aligned}$$

Also using the properties of proportion, we get

$$\begin{aligned} e &= \frac{r_1 + r_0}{r_1 - r_0} \\ \Rightarrow \quad \frac{e}{1} &= \frac{r_1 - r_0}{r_1 + r_0} \\ \Rightarrow \quad \frac{e+1}{e-1} &= \frac{(r_1 + r_0) + (r_1 - r_0)}{(r_1 + r_0) - (r_1 - r_0)} \\ &= \frac{2r_1}{2r_0} \\ &= \frac{r_1}{r_0} \end{aligned}$$

□

Problem 3.8. Mars has a perihelion distance of $204,520,000$ km and an aphelion distance of $246,280,000$ km.

(a) Use these data to calculate the eccentricity.

(b) Find the period of Mars.

(c) Choose a polar coordinate system with the center of the Sun at the pole, and find an equation for the orbit of Mars in that coordinate system.

(d) Use a graphing utility to generate the orbit of Mars from the equation obtained in part (c).

Solution. Given that $r_0 = 204,520,000$ km and $r_1 = 246,280,000$ km.
(a) Orbit of Mars is elliptical. The eccentricity, e is given by

$$e = \frac{r_1 - r_0}{r_1 + r_0} = \frac{246280000 - 204520000}{246280000 + 204520000} = 0.093$$

(b) Period, $T = a^{3/2}$ where a is the semimajor axis. For an ellipse

$$\begin{aligned} a &= \frac{1}{2}(r_1 + r_0) \\ &= \frac{1}{2}(246280000 + 204520000) \\ &= 225400000\,\text{km} \\ &\approx 1.503\,\text{AU} \end{aligned}$$

Hence

$$T = 1.503^{3/2} \approx 1.84\,\text{years}$$

(c) Equation of the orbit of Mars is $r = \frac{a(1-e^2)}{1+e\cos\theta}$ AU. We have

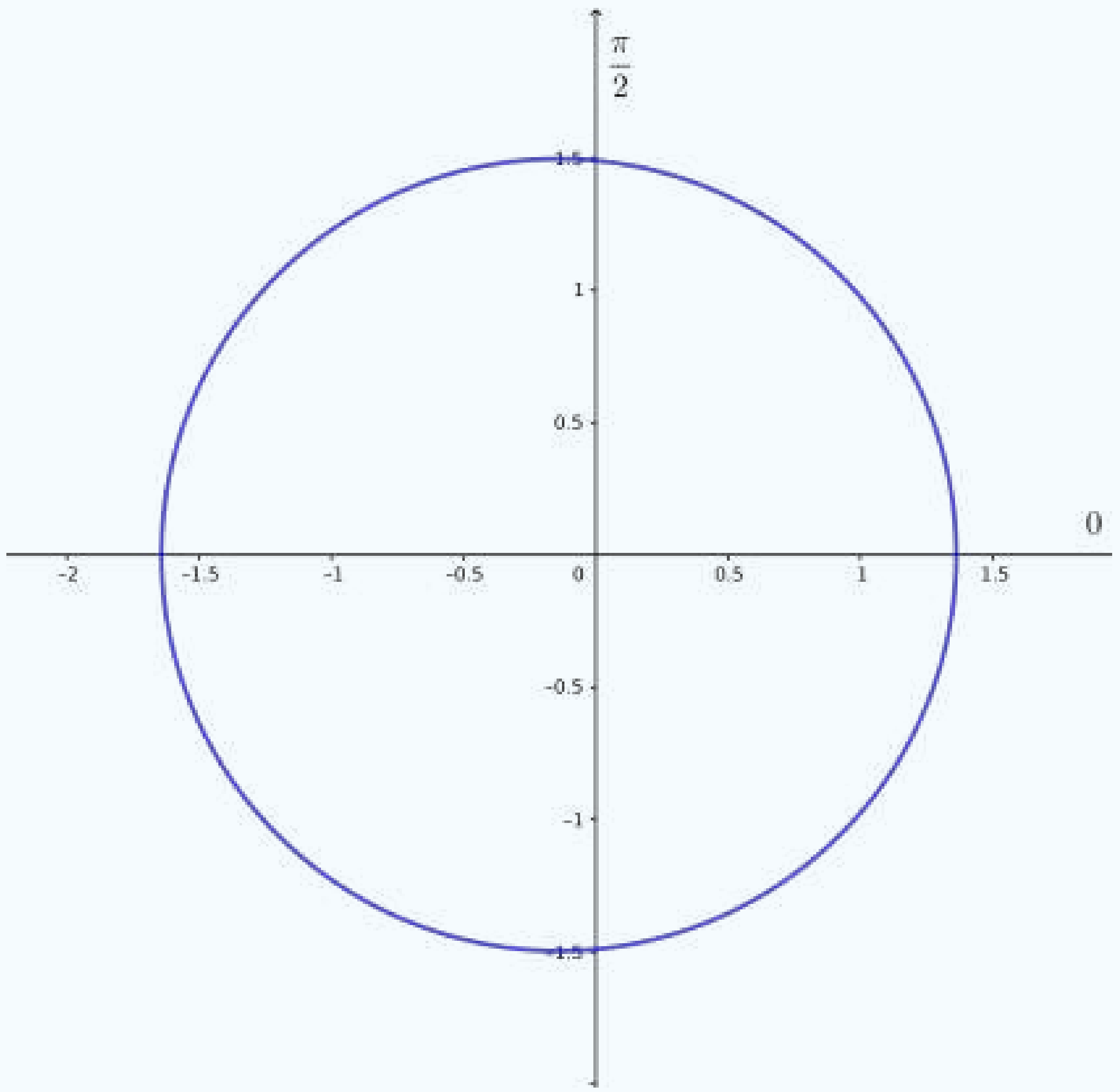
$$a(1 - e^2) = 1.503(1 - 0.093^2) = 1.49$$

Hence

$$r = \frac{1.49}{1 + 0.093\cos\theta}\,\text{AU}.$$

(d)

Graph



□

Problem 3.9. Vanguard 1 was launched in March 1958 into an orbit around the Earth with eccentricity $e = 0.21$ and semimajor axis, $a = 8864.5\,\text{km}$. Find the minimum and maximum heights of Vanguard 1 above the surface of the Earth.

Solution. Radius of earth is 6440 km. We have

$r_0 = a(1-e) = 8864.5(1-0.21) = 7003\,\text{km}$ and $r_1 = a(1+e) = 8864.5(1+0.21) = 10726\,\text{km}$

Hence minimum height above earth's surface is

$$h_{min} = r_0 - 6440 = 7003 - 6440 = 563\,\text{km}$$

and maximum height above earth's surface is

$$h_{max} = r_1 - 6440 = 10726 - 6440 = 4826\,\text{km}$$

□

References

[1] Howard Anton, Irl Bivens, Stephen Davis, *Calculus*, 10^{th} Edition, JOHN WILEY & SONS, INC.